Do stringy corrections stabilize colored black holes?

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We consider hairy black hole solutions of Einstein-Yang-Mills-dilaton theory, coupled to a Gauss-Bonnet curvature term, and we study their stability under small, spacetime-dependent perturbations. We demonstrate that stringy corrections do not remove the sphaleronic instabilities of colored black holes with the number of unstable modes being equal to the number of nodes of the background gauge function. In the gravitational sector and in the limit of an infinitely large horizon, colored black holes are also found to be unstable. Similar behavior is exhibited by magnetically charged black holes while the bulk of neutral black holes are proved to be stable under small, gauge-dependent perturbations. Finally, electrically charged black holes are found to be characterized only by the existence of a gravitational sector of perturbations. As in the case of neutral black holes, we demonstrate that for the bulk of electrically charged black holes no unstable modes arise in this sector.

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I. INTRODUCTION

To date a large number of hairy black hole geometries have been found in a diverse range of matter models coupled to various theories of gravity (for a recent review of some aspects, see [1]). Stable examples of black holes with hair are of particular interest for their physical relevance and also in order to investigate the effects of hair on quantum processes connected with black holes. However, many of the hairy black holes currently known are unstable, particularly those involving non-Abelian gauge fields [2–8], where the instability is topological in nature and similar to that of the flat space sphaleron. The only exceptions to the above rule are the black hole solutions found in the framework of Einstein-Skyrme theory [9] and magnetically charged, non-Abelian black holes in the limit of infinitely strong coupling of the Higgs field [10] or in the presence of a negative cosmological constant [11]. The limited number of stable black holes known so far makes the quest for new stable solutions both immediate and challenging.

The superstring effective action at low energies that follows from the compactification of the heterotic superstring theory provides us with a generalized theory of gravity which is an excellent framework for the study of black holes. The theory contains, apart from the usual Einstein term, a number of scalar fields, the dilaton, axion, and moduli fields, coupled to higher-derivative gravitational terms as well as non-Abelian gauge fields. Here, we are particularly interested in the theory that describes the coupling of the dilaton field to the one-loop Gauss-Bonnet curvature term, and can also include a non-Abelian gauge field. These models are known to possess black hole solutions with hair, both with [12–14] and without [15,16] the gauge field. In the absence of the gauge field, the so-called dilatonic hairy black holes found in [15] were proved to be linearly stable [17] with their stability being in accordance with their interpretation as a generalization of the Schwarzschild black hole in the framework of string theory. This family of stable black holes corresponds to the upper branch of the neutral black hole solutions found later in [13] and whose relative stability with respect to a second unstable branch of solutions was demonstrated by the use of catastrophe theory [18]. However, the question of the stability of the corresponding colored black holes that arise in the presence of the non-Abelian gauge field still remains open. In this article we investigate whether the stability of dilatonic black holes extends to the case where the dilaton is also coupled to the gauge field. We will be particularly interested in stringy colored black holes, and it will be shown that they, like their nonstringy counterparts, are topologically unstable. In addition, conclusions on the stability of dilatonic black holes under small, gauge perturbations as well as that of magnetically and electrically charged black holes will also be drawn.

The outline of the paper is as follows. In Sec. II we introduce our model and briefly review the properties of the stringy black hole solutions. The solutions fall into four categories: neutral, colored, magnetically charged, and electrically charged, depending on the behavior of the gauge field. For the first three types of solutions, the linearized perturbation equations decouple, under an appropriate choice of gauge, into two sectors, corresponding to gravitational and sphaleronic perturbations. We first concentrate on the sphaleronic sector, and show in Sec. III that there are topological instabilities for both colored and magnetically charged black holes. We then count the number of unstable modes in this sector and find that it equals the number of zeros of the gauge field function of the background solution. In the same section, we also consider the stability of the sphaleronic sector of neutral black holes. In Sec. IV, we comment on the gravitational sector for all four types of black holes, appealing to catastrophe theory and continuity arguments. Magnetically charged and colored black holes have instabilities in this sector due to the presence of the non-Abelian gauge field, while the bulk of electrically charged and neutral black

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holes are shown to be stable. Section V is devoted to the conclusions derived from our analysis.

II. STRINGY COLORED BLACK HOLES

We consider the following Lagrangian describing stringy corrections to the SU(2) Einstein-Yang-Mills model [12]:

$$\mathcal{L}=\frac{R}{2}+\frac{1}{4}\partial_\mu\phi\partial^\mu\phi+\frac{\alpha' e^{\phi}}{8g^2}(\beta\mathcal{R}^2_{GB}-F^{a\mu\nu}F^{a}{}_{\mu\nu}), \tag{2.1}$$

where ϕ is the dilaton field and $\mathcal{R}^2_{GB}=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2$ the higher-derivative Gauss-Bonnet term. In Eq. (2.1) we have set the gravitational coupling $\kappa^2=8\pi G$ equal to unity, so that α'/g^2 is the single coupling constant. In effective string theory, which is the situation in which we are interested, the constant β is equal to unity, and from now on we shall fix β to have this value. It is known that, in this case, there are no particlelike solutions to the field equations [19]. In this paper, we are concerned only with the black hole solutions, and shall not consider the particlelike solutions which exist for other values of β. However, much of our analysis applies equally well for all values of β, and in particular to the case $\beta=0$, which corresponds to the Einstein-Yang-Mills-dilaton (EYMD) black holes considered in [3,4]. If the coupling constant is fixed (for numerical computations the value $\alpha'/g^2=1$ is convenient), then there is only one remaining parameter, namely, the event horizon radius r_h.

We consider the general spherically symmetric *Ansatz* for the SU(2) non-Abelian gauge potential [20]:

$$\mathbf{A}=a_0\hat{\tau}_r dt+b\hat{\tau}_r dr+[\nu\hat{\tau}_\theta-(1+w)\hat{\tau}_\varphi]d\theta+[(1+w)\hat{\tau}_\theta+\nu\hat{\tau}_\varphi]\sin\theta\, d\varphi, \tag{2.2}$$

with $\hat{\tau}_r=\hat{\boldsymbol{\tau}}\cdot\boldsymbol{e_r}$, where $\hat{\tau}_i$, $i=1,2,3$, are the usual Pauli matrices. This *Ansatz* for the gauge potential (2.2) does not completely fix the gauge. There is still freedom to make unitary transformations of the form

$$\mathbf{A}\rightarrow\mathcal{T}\mathbf{A}\mathcal{T}^{-1}+\mathcal{T}d\mathcal{T}^{-1}, \tag{2.3}$$

where

$$\mathcal{T}=\exp[k(r,t)\hat{\tau}_r], \tag{2.4}$$

under which the gauge potential components transform according to

$$\begin{pmatrix} a_0\\ b\\ w\\ \nu\end{pmatrix}\rightarrow\begin{pmatrix} a_0-\dot{k}\\ b-k'\\ w\cos k-\nu\sin k\\ \nu\cos k+w\sin k\end{pmatrix}, \tag{2.5}$$

where here, and in the rest of the paper, we use prime to denote the derivative with respect to r, and $\dot{}$ to denote derivative with respect to t. We shall make use of this gauge freedom to choose the gauge for which the perturbation equations take their simplest form.

We also consider the following spherically symmetric line element for the spacetime background:

$$ds^2=-e^{\Gamma}dt^2+e^{\Lambda}dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2). \tag{2.6}$$

In the above two *Ansätze* for the non-Abelian gauge field and the line element, Γ, Λ, a_0, b, ν, and w depend on t and r. For the equilibrium solutions, b and ν vanish identically, and all quantities depend on the coordinate r only. Solutions are found by numerical integration, requiring that there be a regular event horizon at $r=r_h$ and that the geometry be asymptotically flat, with no singularities outside the event horizon. The requirement of asymptotic flatness places the following constraints on the matter fields as $r\rightarrow\infty$: the dilaton $\phi\rightarrow 0$, while the gauge field function $a_0\rightarrow 0$. Finally, $w\rightarrow\pm 1$ for colored black holes with vanishing charge at infinity and $w\rightarrow 0$ for the globally magnetically charged solutions. The types of solutions fall into four categories, depending on the form of the gauge field.

(1) *Neutral* black holes, for which $w\equiv\pm 1$ and $a_0\equiv 0$. These solutions have a vanishing gauge field strength and were discussed in [15,17].

(2) *Magnetically charged* black holes, where $a_0\equiv 0$ and $w\equiv 0$. These have a fixed magnetic charge of unity once the coupling constants are fixed [13].

(3) *Colored* black holes, where $a_0\equiv 0$ and w varies. These are the stringy sphalerons described in [12]. The function w has at least one zero, and the solutions are characterized by the number of nodes of w.

(4) *Electrically charged* black holes, where a_0 does not vanish. The equations describing these black holes are found from the general field equations [derived by varying the action (2.1)] by setting $c\equiv a_0 w\equiv 0$ and then $w\equiv\pm 1$. The form of the function $a_0(r)$ is fixed by the field equations to satisfy

$$a_0'=e^{(\Gamma+\Lambda)/2}e^{-\phi}\frac{Q}{r^2}, \tag{2.7}$$

where the electric charge Q of the black hole can be varied, provided a naked singularity is not formed [13].

The form of the metric and dilaton fields is qualitatively the same for all four types of solutions, with the dilaton field monotonically decreasing to its asymptotic value and the metric functions interpolating between their horizon and asymptotic values at infinity. Regular black hole solutions of each type exist for all values of r_h above a critical value at which point a naked singularity is formed. In the case of dilatonic and colored black holes with a Gauss-Bonnet term the above constraint takes the form [12,15]

$$\frac{\alpha'}{g^2}e^{\phi_h}<\frac{r_h^2}{\sqrt{6}}. \tag{2.8}$$

The first three types of solution are easily studied within the same algebraic framework, simply choosing w as appropriate for each case. Since the electrically charged black holes have a nonzero function a_0, they will have to be considered separately in the stability analysis.

III. SPHALERONIC INSTABILITIES

In this section we consider the magnetically charged and colored black holes and investigate the existence of sphaleronic instabilities under small, bounded, spacetime-dependent perturbations. We use the *temporal gauge* and set $a_0=0$. The advantage of using the temporal gauge is that the system of perturbation equations decouples into two sectors, the *gravitational* sector consisting of $\delta\Gamma$, $\delta\Lambda$, $\delta\phi$, and δw, and the *sphaleronic* sector which comprises δb and $\delta\nu$. For electrically charged black holes, the most useful choice of gauge is not immediately apparent. This will be considered in Sec. IV, where it is shown that electrically charged black holes effectively have only gravitational sector perturbations. Here, we concentrate on the study of the sphaleronic sector of the colored and magnetically charged black holes, leaving the stability analysis of their gravitational sector also for Sec. IV. Our work in this section will also be applicable to the stability analysis of neutral black holes under small, gauge perturbations, and the corresponding conclusions will be drawn simply by setting $w\equiv\pm1$ in the following. The analysis of Sec. III B will then confirm that neutral black holes do not have any instabilities in this sector.

For the sphaleronic sector, and colored and magnetically charged black holes, we have the following perturbation equations, where we have considered periodic perturbations of the form $\delta\mathcal{P}(r,t)=\delta\mathcal{P}(r)e^{i\sigma t}$:

$$\sigma\left[\delta b'+\delta b\left(\phi'+\frac{2}{r}-\frac{\Gamma'}{2}-\frac{\Lambda'}{2}\right)+\frac{2e^{\Lambda}}{r^2}w\;\delta\nu\right]=0 \tag{3.1}$$

and

$$2e^{\Gamma-\Lambda}\begin{pmatrix}\mathcal{H}_{bb} & \mathcal{H}_{b\nu}\\ -\mathcal{H}_{\nu b} & -\mathcal{H}_{\nu\nu}\end{pmatrix}=\sigma^2\begin{pmatrix}r^2e^{-\Lambda} & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}\delta b\\ \delta\nu\end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned}\mathcal{H}_{bb}&=w^2\delta b,\\ \mathcal{H}_{b\nu}&=w\,\delta\nu'-w'\,\delta\nu,\\ \mathcal{H}_{\nu b}&=(w\,\delta b)'+w\left(\phi'+\frac{\Gamma'-\Lambda'}{2}\right)\delta b+w'\,\delta b,\\ \mathcal{H}_{\nu\nu}&=\delta\nu''+\left(\phi'+\frac{\Gamma'-\Lambda'}{2}\right)\delta\nu'+\frac{e^{\Lambda}}{r^2}(1-w^2)\,\delta\nu.\end{aligned} \tag{3.3}$$

These equations depend on the dilaton field ϕ; however, they have no dependence on the Gauss-Bonnet curvature term, except through the static solutions. Therefore, our analysis applies equally well to the case of EYMD black holes. In the following subsections, we shall first of all prove the existence of unstable modes for both magnetically charged and colored black holes, and then proceed to count the number of modes of instability. Our analysis will not depend on the details of the equilibrium solutions, only on properties such as the number of nodes of the gauge field. In this sense the instabilities are ''topological.''

A. Existence of instabilities

In this subsection we shall employ a variational method to show that there are sphaleronic instabilities for both colored and magnetically charged black holes. The sphaleronic sector perturbation equations (3.2) take the form

$$\mathcal{H}\begin{pmatrix}\delta b\\ \delta\nu\end{pmatrix}=\sigma^2\mathcal{A}\begin{pmatrix}\delta b\\ \delta\nu\end{pmatrix}. \tag{3.4}$$

The operator $\mathcal{H}$ must be self-adjoint with respect to a suitable inner product on the space of functions

$$\Psi=\begin{pmatrix}\delta b\\ \delta\nu\end{pmatrix},$$

and the operator $\mathcal{A}$ is required to be positive definite, so that $\langle\Psi|\mathcal{A}|\Psi\rangle>0$ for all nonzero Ψ, using the same inner product. The variational method has been applied successfully to various systems involving non-Abelian gauge fields (see, for example, [7]), and involves defining the following functional:

$$\sigma^2[\Psi]=\frac{\langle\Psi|\mathcal{H}|\Psi\rangle}{\langle\Psi|\mathcal{A}|\Psi\rangle}=\frac{\langle\mathcal{H}\rangle}{\langle\mathcal{A}\rangle} \tag{3.5}$$

for any trial function Ψ. The lowest eigenvalue of the system (3.2) gives a lower bound for this functional. Therefore, there are negative eigenvalues σ^2 (which correspond to unstable modes) if we can find any function Ψ which satisfies $\sigma^2[\Psi]<0$, with $\langle\mathcal{A}\rangle<\infty$. The advantage of this approach is that it is easier to find trial functions which satisfy these criteria than it is to find eigenfunctions, which often involves numerical analysis. The disadvantage is that we do not obtain precise information about the number or magnitude of the negative eigenvalues. In this subsection we are interested in showing that the presence of the dilaton and Gauss-Bonnet term in our model is not sufficient to render the gauge field hair topologically stable. We shall return to the question of the number of unstable modes in the next subsection.

The inner product of two trial functions is defined as

$$\langle\Psi|\Phi\rangle=\int_{r_h}^{\infty}e^{\phi}e^{-(\Gamma-\Lambda)/2}\overline{\Psi}\Phi\,dr, \tag{3.6}$$

and with respect to this inner product the operator $\mathcal{H}$ is self-adjoint, while $\mathcal{A}$ is positive definite. This inner product (3.6) is slightly different from the one usually employed (see, for example, [7]) due to the e^{ϕ} term. This term is crucial if $\mathcal{H}$ is to be self-adjoint. We consider the following trial functions:

$$\begin{aligned}\delta b&=-w'Z_k(r),\\ \delta\nu&=(w^2-1)Z_k(r),\end{aligned} \tag{3.7}$$

which are the same as those used in the Einstein-Yang-Mills-Higgs case [7] [these are appropriate due to our inner product (3.6), which effectively absorbs all the ϕ dependence in this sector]. In order to define the functions Z_k, first introduce a new coordinate ρ:

$$\frac{d\rho}{dr}=e^{\phi}e^{-(\Gamma-\Lambda)/2}. \tag{3.8}$$

This is not the usual ''tortoise'' coordinate because of the ϕ dependence. Then we define the sequence of functions $Z_k(\rho)$ by [21]

$$Z_k(\rho)=Z\left(\frac{\rho}{k}\right), \quad k=1,2,\ldots, \tag{3.9}$$

where $Z(\rho)$ is an even function which is equal to unity for $\rho\in[0,C]$, vanishes for $\rho>C+1$, and satisfies

$$-D\leqslant\frac{dZ}{d\rho}<0 \quad \text{for} \quad \rho\in[C,C+1], \tag{3.10}$$

with C and D arbitrary positive constants.

With these trial functions, after a lengthy calculation, we have

$$\langle\mathcal{A}\rangle=\int_{r_h}^{\infty}dr\, e^{\phi}e^{-(\Gamma-\Lambda)/2}Z_k^2[r^2e^{-\Lambda}w'^2+2(w^2-1)^2],$$

$$\begin{aligned}\langle\mathcal{H}\rangle=&-\int_{r_h}^{\infty}dr\, 2e^{\phi}e^{(\Gamma-\Lambda)/2}\mathcal{J}\\&+\int_{r_h}^{\infty}dr\, 2e^{\phi}e^{(\Gamma-\Lambda)/2}\mathcal{J}(1-Z_k^2)\\&+\int_{r_h}^{\infty}dr\, 2e^{\phi}e^{(\Gamma-\Lambda)/2}(w^2-1)^2Z_k'^2,\end{aligned} \tag{3.11}$$

where

$$\mathcal{J}=w'^2+\frac{e^{\Lambda}}{r^2}(w^2-1)^2\geqslant 0, \tag{3.12}$$

and all boundary terms have vanished due to the definition of the functions Z_k. Immediately it can be seen that the expectation values of $\mathcal{A}$ and $\mathcal{H}$ are finite for each value of k, and the expectation value of $\mathcal{A}$ is positive, as expected. The second and third terms in Eqs. (3.11) converge to zero as $k\to\infty$, and hence choosing k sufficiently large, we obtain a negative expectation value for $\mathcal{H}$. We conclude that the system of perturbation equations (3.2) has at least one negative mode, and hence that the black holes are unstable in the sphaleronic sector.

The other equation in this sector (3.1) is known as the *Gauss constraint*. However, we do not need to consider it here, since it is automatically satisfied by eigenfunctions of the system (3.2). To see this, consider the ''pure gauge'' functions given by

$$\delta b=\mathcal{F}', \quad \delta\nu=-w\mathcal{F}, \tag{3.13}$$

where $\mathcal{F}$ is an arbitrary (differentiable) function of r. The perturbations (3.13) satisfy Eq. (3.2) with eigenvalue $\sigma^2=0$ for any $\mathcal{F}$. Therefore, if $(\delta b,\delta\nu)$ are eigenfunctions with eigenvalue $\sigma^2\neq 0$, they must be orthogonal to the ''pure gauge'' modes:

$$\left\langle\left(\begin{array}{c}\mathcal{F}'\\-w\mathcal{F}\end{array}\right)\middle|\mathcal{A}\middle|\left(\begin{array}{c}\delta b\\\delta\nu\end{array}\right)\right\rangle=0. \tag{3.14}$$

Performing an integration by parts, we obtain

$$\int_{r_h}^{\infty}dr\, e^{\phi}e^{-(\Gamma+\Lambda)/2}r^2\mathcal{F}\mathcal{G}=0, \tag{3.15}$$

where $\mathcal{G}$ is the expression in brackets in the Gauss constraint (3.1). Since $\mathcal{F}$ is arbitrary and the integrand is assumed to be continuous, the Gauss constraint must be satisfied by the perturbations δb and $\delta\nu$.

We note that the analysis of this subsection applies equally well to the colored black holes of [12] as well as the magnetically charged black holes which occur when $w\equiv 0$ [13]. The instability of the magnetically charged black holes may be understood by considering them as colored black holes in the limit in which the number of nodes of the gauge field function w goes to infinity. Therefore, they correspond to saddle points of the action (2.1) due to the fact that the gauge group is non-Abelian. In addition, since the sphaleronic sector perturbation equations do not depend on the Gauss-Bonnet term and the proof of instability depends not on the details of the equilibrium solutions, but only on their global features (such as the existence of the event horizon), the analysis also applies to EYMD black holes [3]. We emphasize that the calculations necessary to obtain the expectation value (3.11) made use only of the static equilibrium equation for w (which has the same form for both the Gauss-Bonnet and EYMD models) and not that for ϕ, which does involve the Gauss-Bonnet term. Therefore EYMD black holes are also unstable in the sphaleronic sector. This is what would be anticipated, but previous studies of the stability of the EYMD system have concentrated only on the gravitational sector [3]. Therefore the presence of the dilaton (either with or without an associated Gauss-Bonnet term) is not sufficient to remove the topological instabilities of the Yang-Mills field. So far, the only way to do this is to introduce a negative cosmological constant [11].

B. Counting the number of unstable modes

Having shown that there exist instabilities in the sphaleronic sector, we now count the number of unstable modes. This subsection extends the method of [6], which counts the number of sphaleronic instabilities of Einstein-Yang-Mills (EYM) black holes, to systems involving a dilaton field and Gauss-Bonnet curvature term.

First, we define the usual ''tortoise'' coordinate r^* [compare ρ, Eq. (3.8)],

$$\frac{dr^*}{dr}=e^{-(\Gamma-\Lambda)/2}, \tag{3.16}$$

and introduce the quantities

$$\chi=\frac{r^2}{2}e^{-(\Gamma+\Lambda)/2}e^{\phi}\delta b, \quad \gamma^2=\frac{2e^{\Gamma}}{r^2}. \tag{3.17}$$

Then the perturbed equations for the sphaleronic sector (3.2) can be written in the more compact form

$$0=\frac{d\chi}{dr^*}+e^{\phi}w\,\delta\nu,$$

$$\sigma^2\chi=w^2\gamma^2\chi+e^{\phi}\left(w\frac{d}{dr^*}\delta\nu-\frac{dw}{dr^*}\delta\nu\right), \tag{3.18}$$

$$-\sigma^2e^{\phi}w\,\delta\nu=\frac{d}{dr^*}\left[w^2\gamma^2\chi+e^{\phi}\left(w\frac{d}{dr^*}\delta\nu-\frac{dw}{dr^*}\delta\nu\right)\right].$$

The first of these equations is the Gauss constraint which we assume to hold even in the case that $\sigma=0$. In that case, we can easily prove that the third equation is a direct consequence of the first two and it will be ignored hereafter. In terms of the new function

$$u=\frac{e^{-\phi/2}\chi}{w}, \tag{3.19}$$

the first two perturbation equations in Eqs. (3.18) can be rearranged to give

$$-\frac{d^2u}{dr^{*2}}+\mathcal{U}_1(r^*)u=\sigma^2u, \tag{3.20}$$

where

$$\mathcal{U}_1(r^*)=\frac{1}{2}\gamma^2(1+w^2)+\frac{2}{w^2}\left(\frac{dw}{dr^*}\right)^2+\frac{2}{w}\frac{dw}{dr^*}\frac{d\phi}{dr^*}+\frac{1}{4}\left(\frac{d\phi}{dr^*}\right)^2-\frac{1}{2}\frac{d^2\phi}{dr^{*2}}. \tag{3.21}$$

This is a standard Schrödinger equation, but the potential $\mathcal{U}_1$ is not regular because w has zeros. Therefore we need to use the method of [6] to map this equation to a ''dual'' Schrödinger equation which will have a regular potential.

The ''pure gauge'' modes (3.13) give a solution of Eq. (3.20) when $\sigma=0$, namely,

$$u_0=\frac{e^{\phi/2}}{w\gamma^2}\frac{d\mathcal{F}}{dr^*}, \tag{3.22}$$

where in order that the Gauss constraint be satisfied (even though $\sigma=0$) the arbitrary function $\mathcal{F}$ must satisfy the differential equation

$$\frac{d}{dr^*}\left(\frac{e^{\phi}}{\gamma^2}\frac{d\mathcal{F}}{dr^*}\right)=e^{\phi}w^2\mathcal{F}. \tag{3.23}$$

The function u_0 can be used to factorize the operator acting on u in Eq. (3.20):

$$Q^+Q^-=-\left(\frac{d}{dr^*}+\frac{1}{u_0}\frac{du_0}{dr^*}\right)\left(\frac{d}{dr^*}-\frac{1}{u_0}\frac{du_0}{dr^*}\right)=-\frac{d^2}{dr^{*2}}+\frac{1}{u_0}\frac{d^2u_0}{dr^{*2}}. \tag{3.24}$$

By defining $\psi=Q^-u$ and applying Q^- again, we obtain the ''dual'' eigenvalue equation

$$Q^-Q^+\psi=-\frac{d^2\psi}{dr^{*2}}+\mathcal{U}_2(r^*)\psi=\sigma^2\psi, \tag{3.25}$$

where

$$\mathcal{U}_2(r^*)=\frac{1}{2}\gamma^2(3w^2-1)+2\frac{d}{dr^*}(w^2Y)+\frac{1}{2}\frac{d^2\phi}{dr^{*2}}+\frac{1}{4}\left(\frac{d\phi}{dr^*}\right)^2. \tag{3.26}$$

In the above, Y is a function of r^* defined as

$$Y=-\gamma^2\mathcal{F}\left(\frac{d\mathcal{F}}{dr^*}\right)^{-1} \tag{3.27}$$

and satisfying the equation

$$\frac{dY}{dr^*}=-\gamma^2+w^2Y^2+\frac{d\phi}{dr^*}Y. \tag{3.28}$$

Equation (3.25) is another standard Schrödinger equation, but now we have a potential which is regular everywhere outside the event horizon, provided that Y is a regular, well-defined, bounded function for all r^*. Assuming for the moment that this is the case (we shall return to this issue shortly), the solution ψ_0 of Eq. (3.25) when $\sigma=0$ will then have the same number of zeros as there are eigenfunctions with $\sigma^2<0$, that is, the number of unstable modes. The zero energy solution ψ_0 satisfies the equation

$$Q^+\psi_0=\left(-\frac{d}{dr^*}-\frac{1}{u_0}\frac{du_0}{dr^*}\right)\psi_0=\left(-\frac{d}{dr^*}+w^2Y+\frac{1}{w}\frac{dw}{dr^*}+\frac{1}{2}\frac{d\phi}{dr^*}\right)\psi_0=0, \tag{3.29}$$

leading to the solution

$$\psi_0 = w\, e^{\phi/2} \exp\left(\int_0^{r^*} w^2 Y\, dr^* \right). \tag{3.30}$$

The function ψ_0 then has the same number of zeros as w if the last factor on the right-hand side of Eq. (3.30) is regular for all r^*. Assuming, in addition, that ψ_0 is normalizable, this leads us to conclude that the number of unstable modes in the sphaleronic sector equals the number of zeros of the gauge field function w for the colored stringy black holes.

At this stage we need to check our assumptions, namely, that the function Y, satisfying the differential equation (3.28), is regular for all r^* and vanishes sufficiently quickly in the asymptotic regimes $r^* \to \pm\infty$ so that the integral in Eq. (3.30) is finite everywhere. We also require that the function ψ_0 given by Eq. (3.30) be a normalizable eigenfunction. First, from the definition (3.27) of Y, it can be seen that Y will be a regular function as long as $\mathcal{F}$ is regular and $d\mathcal{F}/dr^* \neq 0$. The behavior of $\mathcal{F}$ is most easily found by considering the differential equation (3.23) in terms of the radial coordinate r:

$$\mathcal{F}'' + \left(\phi' + \frac{2}{r} - \frac{\Gamma'}{2} - \frac{\Lambda'}{2} \right) \mathcal{F}' - \frac{2 e^{\Lambda} w^2}{r^2} \mathcal{F} = 0. \tag{3.31}$$

This equation has regular singular points at $r = r_h, \infty$. Near $r = r_h$, the standard Frobenius method reveals that either $\mathcal{F} \sim O(1)$ or $\mathcal{F} \sim O(r - r_h)$ as $r \to r_h$. We consider the solution of Eq. (3.31) which has the behavior $\mathcal{F} \sim (r - r_h)$. Since Eq. (3.31) has only one other singular point, at infinity, this solution can be extended to a solution regular for all r. Then, as $r \to \infty$, applying the Frobenius method again shows that either $\mathcal{F} \sim O(r)$ or $\mathcal{F} \sim O(r^{-2})$. It must be the case that $\mathcal{F} \sim r$, since we can show that $\mathcal{F}$ cannot vanish at both $r = r_h$ and as $r \to \infty$, as follows. From Eq. (3.23) we have

$$0 \leq \int_{r_0^*}^{r_1^*} e^{\phi} w^2 \mathcal{F}^2 dr^* = \int_{r_0^*}^{r_1^*} \mathcal{F} \frac{d}{dr^*} \left(\frac{e^{\phi}}{\gamma^2} \frac{d\mathcal{F}}{dr^*} \right) dr^*$$
$$= \left[\frac{\mathcal{F} e^{\phi}}{\gamma^2} \frac{d\mathcal{F}}{dr^*} \right]_{r_0^*}^{r_1^*} - \int_{r_0^*}^{r_1^*} \frac{e^{\phi}}{\gamma^2} \left(\frac{d\mathcal{F}}{dr^*} \right)^2 dr^*. \tag{3.32}$$

Taking $r_0^* \to -\infty$ (corresponding to $r \to r_h$), $r_1^* \to \infty$ ($r \to \infty$), we obtain a contradiction if $\mathcal{F} \to 0$ for both limits (if $\mathcal{F}$ is not identically zero). Similarly, if $d\mathcal{F}/dr^* = 0$ for $r^* = r_1^*$ and taking $r_0^* \to -\infty$, we also obtain a contradiction, which means that $d\mathcal{F}/dr^*$ cannot be zero for $r^* \in (-\infty, \infty)$. Therefore Y is a regular function of r^* for all r^*. In addition, the differential equation (3.28) shows that

$$Y \sim -\frac{1}{w_h^2 r^*} \to 0 \quad \text{as} \quad r^* \to -\infty,$$

$$Y \sim -\frac{1}{r^*} \to 0 \quad \text{as} \quad r^* \to \infty. \tag{3.33}$$

This means that $|\psi_0|^2 \sim (r^*)^{-2}$ as $r^* \to \pm\infty$, and so ψ_0 is a normalizable wave function. Therefore the conditions necessary for our conclusion—that there are as many negative modes as zeros of the function w—to hold are satisfied.

The analysis of this subsection once again reveals the topological nature of the instabilities, since the precise details of the equilibrium solutions were not needed, but only general properties such as the behavior in the asymptotic regions and the number of nodes of the gauge function. In particular, the only equilibrium field equation used in the calculations is that for w, which does not depend explicitly on the Gauss-Bonnet term. Therefore, the result holds for all values of β, so both the colored black holes with a Gauss-Bonnet curvature term and the EYMD solutions are covered. The number of unstable modes is exactly the same as for EYM black holes [6], so the stringy corrections make no difference to the topological instabilities.

So far in this subsection we have been concerned with colored black holes, since we have assumed implicitly that w does not vanish identically. For magnetically charged black holes, where $w \equiv 0$, the sphaleronic sector perturbation equations (3.2) reduce to $\delta b \equiv 0$ (implying that the Gauss constraint is satisfied) and

$$\sigma^2 \delta\nu + \frac{d^2}{dr^{*2}} \delta\nu + \frac{d\phi}{dr^*} \frac{d}{dr^*} \delta\nu + \frac{e^{\Gamma}}{r^2} \delta\nu = 0. \tag{3.34}$$

This can be cast into the form of a standard Schrödinger equation for the variable $\xi = e^{\phi/2} \delta\nu$:

$$\sigma^2 \xi + \frac{d^2 \xi}{dr^{*2}} - U_S(r^*) \xi = 0, \tag{3.35}$$

where the potential is

$$U_S(r^*) = \frac{1}{2} \frac{d^2\phi}{dr^{*2}} + \frac{1}{4} \left(\frac{d\phi}{dr^*} \right)^2 - \frac{e^{\Gamma}}{r^2}. \tag{3.36}$$

In the next section we shall find that a Schrödinger equation with this potential also governs the gravitational sector perturbations of the magnetically charged black holes. We shall be able to appeal to catastrophe theory to show that this equation (3.35) has an infinite number of unstable modes, so that the magnetically charged black holes have infinitely many sphaleronic instabilities. This result was anticipated from regarding the magnetically charged black holes as the limit of colored black holes in which the number of zeros of w goes to infinity.

As we mentioned in the beginning of this section, our analysis is also applicable to the stability analysis of the sphaleronic sector of neutral black holes. Although the corresponding background solutions have no gauge field, a sphaleronic sector arises when one applies small gauge perturbations to the system. From Eq. (3.30), we can easily see that, in this case, the function ψ_0 has no nodes since, by definition, $w^2 \equiv 1$ everywhere. This confirms the absence of unstable modes in the sphaleronic sector of neutral black holes and demonstrates the stability of this family of solutions even under gauge perturbations.

IV. GRAVITATIONAL SECTOR PERTURBATIONS

We now turn to the gravitational sector perturbations. The perturbation equations in this sector, as shall be seen below, are extremely unattractive (in [17] a great deal of complex numerical work was necessary) and so we shall appeal to the catastrophe theory analysis of [13]. In [13] the authors investigated the properties of the static solutions with varying horizon radius r_h, fixing the values of the coupling constants α'/g^2 and β. In this section we shall first consider the gravitational sector of the colored and magnetically charged black holes. Then, we shall confirm that the addition of the perturbation δw in the gravitational sector of the neutral black holes does not introduce any instabilities. Finally, we will examine the stability of the electrically charged black holes, which need to be studied separately because $a_0 \neq 0$ for the equilibrium solutions.

A. Colored and magnetically charged black holes

The gravitational sector perturbation equations for the neutral, colored, and magnetically charged black holes using the temporal gauge take the form

$$0=\delta\phi''+\delta\phi'\left(\frac{\Gamma'}{2}-\frac{\Lambda'}{2}+\frac{2}{r}\right)-\delta\phi\left[\phi''+\phi'\left(\frac{\Gamma'}{2}-\frac{\Lambda'}{2}+\frac{2}{r}\right)\right]-e^{\Lambda-\Gamma}\delta\ddot{\phi}+\frac{\alpha' e^{\phi}}{g^2r^2}(1-e^{-\Lambda})(e^{\Lambda-\Gamma}\delta\ddot{\Lambda}-\delta\Gamma'')$$
$$+\delta\Gamma'\left\{\frac{\phi'}{2}-\frac{\alpha' e^{\phi}}{g^2r^2}\left[\Lambda' e^{-\Lambda}+(1-e^{-\Lambda})\left(\Gamma'-\frac{\Lambda'}{2}\right)\right]\right\}+\frac{2\alpha' e^{\phi}}{g^2r^2}\left\{\frac{e^{\Lambda}}{r^2}(1-w^2)w\,\delta w-w'\,\delta w'\right\}$$
$$+\delta\Lambda\frac{\alpha' e^{\phi}}{g^2r^2}\left\{\Gamma'\Lambda' e^{-\Lambda}-e^{-\Lambda}\left[\Gamma''+\frac{\Gamma'}{2}(\Gamma'-\Lambda')\right]-\frac{e^{\Lambda}}{2r^2}(1-w^2)^2\right\}-\delta\Lambda'\left\{\frac{\phi'}{2}-\frac{\alpha' e^{\phi}}{g^2r^2}\frac{\Gamma'}{2}(1-3e^{-\Lambda})\right\}, \tag{4.1a}$$

$$0=-e^{\Lambda-\Gamma}\delta\ddot{w}+\delta w''+\delta w'\left(\phi'+\frac{\Gamma'-\Lambda'}{2}\right)+w'\left(\delta\phi'+\frac{\delta\Gamma'-\delta\Lambda'}{2}\right)+\frac{e^{\Lambda}}{r^2}[\delta\Lambda\, w(1-w^2)+\delta w(1-3w^2)], \tag{4.1b}$$

$$0=\delta\Lambda'\left[1+\frac{\alpha' e^{\phi}\phi'}{2g^2r}(1-3e^{-\Lambda})\right]+\delta\phi'\left[-\frac{r\phi'}{2}+\frac{\alpha' e^{\phi}}{2g^2r}\Lambda'(1-3e^{-\Lambda})-\frac{2\alpha' e^{\phi}}{g^2r}\phi'(1-e^{-\Lambda})\right]$$
$$-\delta\phi''\frac{\alpha' e^{\phi}}{g^2r}(1-e^{-\Lambda})+\delta\Lambda\left\{\frac{e^{\Lambda}}{r}+\frac{\alpha' e^{\phi}}{g^2r}\left[e^{-\Lambda}\frac{3\phi'\Lambda'}{2}-e^{-\Lambda}(\phi''+\phi'^2)-\frac{e^{\Lambda}}{4r^2}(1-w^2)^2\right]\right\}$$
$$+\frac{\alpha' e^{\phi}}{g^2r}\left[\frac{e^{\Lambda}}{r^2}(1-w^2)w\,\delta w-w'\,\delta w'\right]+\delta\phi\frac{\alpha' e^{\phi}}{g^2r}\left[\frac{\phi'\Lambda'}{2}(1-3e^{-\Lambda})\right.$$
$$\left.-(1-e^{-\Lambda})(\phi''+\phi'^2)-\frac{e^{\Lambda}}{4r^2}(1-w^2)^2-\frac{w'^2}{2}\right], \tag{4.1c}$$

$$0=\delta\Gamma'\left[1+\frac{\alpha' e^{\phi}\phi'}{2g^2r}(1-3e^{-\Lambda})\right]+\delta\phi'\left[-\frac{r\phi'}{2}+\frac{\alpha' e^{\phi}}{2g^2r}\Gamma'(1-3e^{-\Lambda})\right]-\delta\ddot{\phi}\,\frac{\alpha' e^{\phi}}{g^2r}e^{\Lambda-\Gamma}(1-e^{-\Lambda})$$
$$+\delta\phi\frac{\alpha' e^{\phi}}{2g^2r}\left[\phi'\Gamma'(1-3e^{-\Lambda})+\frac{e^{\Lambda}}{2r^2}(1-w^2)^2-w'^2\right]-\frac{\alpha' e^{\phi}}{g^2r}\left[\frac{e^{\Lambda}}{r^2}(1-w^2)w\,\delta w+w'\,\delta w'\right]$$
$$+\delta\Lambda\left\{-\frac{e^{\Lambda}}{r}+\frac{\alpha' e^{\phi}}{2g^2r}\left[3\phi'\Lambda' e^{-\Lambda}+\frac{e^{\Lambda}}{2r^2}(1-w^2)^2\right]\right\}, \tag{4.1d}$$

$$0=\delta\dot{\Lambda}\left[1+\frac{\alpha' e^{\phi}\phi'}{2g^2r}(1-3e^{-\Lambda})\right]-\frac{r\phi'}{2}\delta\dot{\phi}-\frac{\alpha' e^{\phi}}{g^2r}\left[(1-e^{-\Lambda})\left(\delta\dot{\phi}\phi'+\delta\bar{\phi}'-\delta\dot{\phi}\frac{\Gamma'}{2}\right)+w'\,\delta\dot{w}\right], \tag{4.1e}$$

$$0=\delta\Gamma''\left(1-\frac{\alpha' e^{\phi-\Lambda}}{g^2 r}\phi'\right)+\frac{\delta\Gamma'}{2}(\Gamma'-\Lambda')+(\delta\Gamma'-\delta\Lambda')\left(\frac{\Gamma'}{2}+\frac{1}{r}\right)+\phi'\,\delta\phi'-e^{\Lambda-\Gamma}\delta\ddot{\Lambda}-\frac{\alpha' e^{\phi-\Lambda}}{g^2 r}$$
$$\times\left\{(\delta\phi''+2\phi'\,\delta\phi')\Gamma'+\delta\phi'\Gamma''+\delta\Gamma'(\phi''+\phi'^2)+\left(\frac{\delta\phi'\Gamma'}{2}+\frac{\phi'\,\delta\Gamma'}{2}\right)(\Gamma'-3\Lambda')+(\delta\phi-\delta\Lambda)\right.$$
$$\left.\times\left[\phi'\Gamma''+\Gamma'(\phi''+\phi'^2)+\frac{\phi'\Gamma'}{2}(\Gamma'-3\Lambda')\right]+\frac{\phi'\Gamma'}{2}(\delta\Gamma'-3\,\delta\Lambda')+e^{\Lambda-\Gamma}(\Lambda'\,\delta\ddot{\phi}-\phi'\,\delta\ddot{\Lambda})\right\}$$
$$-\frac{\alpha' e^{\phi+\Lambda}}{2g^2 r^4}[(\delta\phi+\delta\Lambda)(1-w^2)^2-4(1-w^2)w\,\delta w]. \tag{4.1f}$$

The presence of the Gauss-Bonnet term means that these perturbation equations are considerably more complex than those of the EYMD system [3]. However, the gauge field does not make them particularly more complicated than for neutral black holes with the Gauss-Bonnet term [17].

We can immediately reduce the number of equations by 1, by integrating Eq. (4.1e) with respect to time to give

$$0=\delta\Lambda\left[1+\frac{\alpha' e^{\phi}\phi'}{2g^2 r}(1-3e^{-\Lambda})\right]-\frac{r\phi'}{2}\delta\phi$$
$$-\frac{\alpha' e^{\phi}}{g^2 r}\left[(1-e^{-\Lambda})\left(\phi'\,\delta\phi+\delta\phi'-\delta\phi\frac{\Gamma'}{2}\right)+w'\,\delta w\right]$$
$$-\mu(r), \tag{4.2}$$

where $\mu(r)$ is an arbitrary function of r. Differentiating the above equation with respect to r and comparing it with Eq. (4.1c) by using, at the same time, the time-independent equations of motion, a lengthy calculation gives the following differential equation for $\mu(r)$:

$$\mu'(r)+\mu(r)\left[\frac{1}{2}(\Gamma'-\Lambda')+\frac{1}{r}\right]=0. \tag{4.3}$$

When integrated, this equation yields $\mu(r)\propto e^{(\Lambda-\Gamma)/2}/r$, which goes to infinity when $r\to r_h$, in contradiction to our assumption of small, bounded perturbations. As a result, the only acceptable solution for the function $\mu(r)$ is the trivial one, $\mu\equiv 0$.

For both colored and magnetically charged black holes, the behavior of the static solutions as r_h is varied is qualitatively the same. Static configurations which solve the field equations exist for all r_h above a certain value, at which point a naked singularity forms. This is in accordance with the result that there are no particlelike solutions in this theory [19]. There is no critical point in the graph of black hole mass against horizon radius [13], and therefore catastrophe theory tells us that the stability of the solutions does not alter as we vary r_h. This result can be considered from another perspective.

The system of equations (4.1) could be rewritten as two coupled linear equations for the perturbations $\delta\phi$ and δw by eliminating $\delta\Lambda$ and $\delta\Gamma$. (This would be a long and tedious calculation; see [17] for the corresponding computation in the neutral case.) Assuming that the equilibrium solutions are analytic in the parameter r_h, it would be possible to invoke functional analysis theorems to show that the negative eigenvalues σ^2 of the system must also be analytic in r_h. Therefore the number of negative eigenvalues (corresponding to the number of unstable modes) can only change when an eigenvalue passes through the value zero. A perturbation changes the mass of the black hole by an amount proportional to the eigenvalue. Therefore a zero mode does not change the mass of the black hole. Furthermore, a zero mode corresponds to a time-independent perturbation, in other words, a small, static perturbation of the equilibrium solutions. Such a perturbation exists only if there are two equilibrium solutions arbitrarily close together for the same value of the black hole mass, that is, at a critical point. Therefore, when there is no critical point (as is the case for colored and magnetically charged black holes), the stability does not change as we vary the parameter r_h. For neutral black holes, there are two branches of solutions [13], the upper branch extending to arbitrarily large r_h. The analysis below would then apply to this upper branch, and we shall below obtain agreement with the known result that this branch of solutions is linearly stable [17].

Since there is no upper bound on the value of r_h, we shall consider the perturbation equations as $r_h\to\infty$, in which case they will take a particularly simple form. From this we shall be able to appeal to the catastrophe theory analysis to deduce the stability of the black holes, whatever their horizon radius. First, introduce the following dimensionless variables:

$$\hat{r}=\frac{r}{r_h},\qquad \hat{t}=\frac{t}{r_h}. \tag{4.4}$$

In that case, all the terms proportional to α' become of $O(1/r_h^2)$ while all the other terms are of $O(1)$. However, we cannot yet take the limit $r_h\to\infty$ since the Gauss-Bonnet terms contain derivatives of the metric functions that diverge at the limit $r\to r_h$. In order to resolve this, we use, once again, the tortoise coordinate transformation (3.16) which now connects $\hat{r}$ with $\hat{r}^*$. Then, the perturbed equations take the form

$$0 = -\delta\ddot{\phi} + \frac{d^2}{d\hat{r}^{*2}}\delta\phi + \frac{2}{\hat{r}}e^{(\Gamma-\Lambda)/2}\frac{d}{d\hat{r}^*}\delta\phi - \left[\frac{d^2\phi}{d\hat{r}^{*2}} + \frac{2}{\hat{r}}e^{(\Gamma-\Lambda)/2}\frac{d\phi}{d\hat{r}^*}\right]\delta\phi + \frac{1}{2}\frac{d\phi}{d\hat{r}^*}\left(\frac{d}{d\hat{r}^*}\delta\Gamma - \frac{d}{d\hat{r}^*}\delta\Lambda\right) + O\left(\frac{1}{r_h^2}\right),$$

$$0 = -\delta\ddot{w} + \frac{d}{d\hat{r}^{*2}}\delta w + \frac{d\phi}{d\hat{r}^*}\frac{d}{d\hat{r}^*}\delta w + \left[\frac{d}{d\hat{r}^*}\delta\phi + \frac{1}{2}\left(\frac{d}{d\hat{r}^*}\delta\Gamma - \frac{d}{d\hat{r}^*}\delta\Lambda\right)\right]\frac{dw}{d\hat{r}^*} + \frac{e^\Gamma}{\hat{r}^2}[w(1-w^2)\delta\Lambda + (1-3w^2)\delta w], \tag{4.5}$$

$$0 = e^{(\Gamma-\Lambda)/2}\frac{d}{d\hat{r}^*}\delta\Gamma - \frac{\hat{r}}{2}\frac{d\phi}{d\hat{r}^*}\frac{d}{d\hat{r}^*}\delta\phi - \frac{e^\Gamma}{\hat{r}}\delta\Lambda + O\left(\frac{1}{r_h^2}\right),$$

$$0 = \delta\Lambda - \frac{\hat{r}}{2}\frac{d\phi}{d\hat{r}^*}\delta\phi + O\left(\frac{1}{r_h^2}\right).$$

As $r_h \to \infty$, in accordance with the no-hair theorem, the geometry becomes that of a Schwarzschild black hole with $\phi \equiv$ const and the Yang-Mills field superimposed on this background. As a result, both of the perturbations $\delta\Lambda$ and $\delta\Gamma$ vanish while the equation for $\delta\phi$ reduces to the form

$$-\frac{d^2\lambda}{d\hat{r}^{*2}} + \frac{e^{(\Gamma-\Lambda)/2}}{\hat{r}}\left(\frac{d\Gamma}{d\hat{r}^*} - \frac{d\Lambda}{d\hat{r}^*}\right)\lambda = \sigma^2\lambda, \tag{4.6}$$

where

$$\lambda = \delta\phi \exp\left(\int_{-\infty}^{\hat{r}^*}\frac{e^{(\Gamma-\Lambda)/2}}{\hat{r}}d\hat{r}^*\right) \tag{4.7}$$

and we have considered periodic perturbations. The above equation takes the form of a Schrödinger-like differential equation with a potential which is everywhere regular as well as positive and vanishes as $\hat{r}^* \to \pm\infty$. We thus may conclude that the subsector of the dilaton and metric perturbations is still characterized by the absence of any unstable modes. Now we turn to the perturbation equation of the Yang-Mills function. By implementing the results derived from the above analysis, this takes the form

$$-\frac{d}{d\hat{r}^{*2}}\delta w - \frac{e^\Gamma}{\hat{r}^2}(1-3w^2)\delta w = \sigma^2\delta w. \tag{4.8}$$

This equation is exactly the same, to leading order in $1/r_h^2$, as the perturbation equation for the gauge field in Einstein-Yang-Mills theory without a dilaton or Gauss-Bonnet term. This was to be expected since the EYM equations also decouple in the limit $r_h \to \infty$ to give a Schwarzschild geometry with a gauge field on this background. It is known that the EYM system possesses instabilities in this sector (see, for example, [2]). The above equation for δw is in agreement with this result. As the background gauge function w oscillates around its zero value [12], a potential well is formed in the region where $w^2 < 1/3$, thus leading to the existence of bound states in the corresponding Schrödinger equation. We therefore conclude that the stringy colored black holes are also unstable and their instabilities are, once again, associated with the existence of nodes of the background gauge function w. It is worth noting that, in the case of colored black holes arising in the presence of a negative cosmological constant in the theory, solutions with a zero number of nodes do exist and they were proved to be linearly stable in both perturbation sectors [11].

That the magnetically charged black holes also possess unstable modes in this sector may be deduced, as in the sphaleronic sector, from the fact that they are the limit of colored black holes in which the number of zeros of the gauge field goes to infinity. For magnetically charged black holes, which follow if we set $w=0$, the potential in the Schrödinger equation (4.8) is everywhere negative and goes to zero as $\hat{r}^* \to \pm\infty$:

$$U(\hat{r}^*) = -\frac{e^\Gamma}{\hat{r}^2}. \tag{4.9}$$

The standard estimate for the number of bound states, $(1/\pi)\int_{-\infty}^{\infty}\sqrt{-U(\hat{r}^*)}d\hat{r}^*$ [22] then shows that there are an infinite number of unstable modes in this case.

This analysis for a very large horizon radius has confirmed what might have been anticipated, namely, that the instabilities are due to the presence of the gauge field. In addition, it is known that EYMD black holes possess gravitational instabilities [3], which is in agreement with our results, assuming that stability does not change as β varies. This is a reasonable assumption, since the behavior of the field equations as $r_h \to \infty$ is the same for all β.

As explained earlier in this subsection, catastrophe theory tells us that the stability of the black holes does not change as we vary r_h. The results above, derived in the limit $r_h \to \infty$, can therefore be extended to arbitrary r_h for which black hole solutions exist. In particular, colored black holes will be unstable in this sector, and magnetically charged black holes will have an infinite number of unstable gravitational modes.

We now exploit the result that magnetically charged black holes have infinitely many unstable modes in the gravitational sector, irrespective of the value of the horizon radius, to infer that they also have infinitely many unstable modes in the sphaleronic sector. For magnetically charged black holes and any value of the horizon radius r_h, the perturbation equation for δw, Eq. (4.1b), decouples from the other equations in the gravitational sector and has the simplified form

$$\sigma^2\xi + \frac{d^2\xi}{dr^{*2}} - U_G(r^*)\xi = 0, \tag{4.10}$$

where $\xi = e^{\phi/2}\delta w$ and

$$U_G(r^*) = \frac{1}{2}\frac{d^2\phi}{dr^{*2}} + \frac{1}{4}\left(\frac{d\phi}{dr^*}\right)^2 - \frac{e^\Gamma}{r^2}. \tag{4.11}$$

This potential is exactly the same as that arising in the sphaleronic sector perturbations of magnetically charged black holes (3.36). Therefore, since the Schrödinger equation (4.10) has infinitely many negative eigenvalues, magnetically charged black holes also have infinitely many unstable modes in the sphaleronic sector, as asserted in the previous section.

For neutral black holes, in the limit of a large horizon radius, the potential in the perturbation equation for δw, Eq. (4.8), after setting $w = \pm 1$, reduces to

$$U(\hat{r}^*) = \frac{2e^\Gamma}{\hat{r}^2}. \tag{4.12}$$

Since this potential is everywhere positive and tends to zero in the asymptotic regions, we can conclude that there are no unstable modes. Therefore neutral black holes are stable as we vary r_h (we emphasize that this applies only to the upper branch of solutions in [13], namely, the branch of solutions which extends to an arbitrarily large horizon radius and which includes the bulk of the background black hole solutions [18]). In the next subsection we shall show, by another method, that the equation for δw and general r_h has no unstable modes. This result will be useful in studying the stability of electrically charged black holes, to which we now turn.

B. Electrically charged black holes

We now turn to electrically charged black holes, which need to be considered separately from the other cases because $a_0 \neq 0$ for the equilibrium solutions. As in the previous subsection, we first consider the linearized perturbation equations involving the metric perturbation $\delta\Lambda$:

$$\begin{aligned}0 = {} & \delta\Lambda'\left[1 + \frac{\alpha' e^\phi \phi'}{2g^2 r}(1 - 3e^{-\Lambda})\right] + \delta\phi'\left[-\frac{r\phi'}{2} + \frac{\alpha' e^\phi}{2g^2 r}\Lambda'(1 - 3e^{-\Lambda}) - \frac{2\alpha' e^\phi}{g^2 r}\phi'(1 - e^{-\Lambda})\right] \\ & + \delta\Lambda\left\{\frac{e^\Lambda}{r} + \frac{\alpha' e^\phi}{g^2 r}\left[e^{-\Lambda}\frac{3\phi'\Lambda'}{2} - e^{-\Lambda}(\phi'' + \phi'^2)\right]\right\} - \frac{\alpha' r}{4g^2}e^\phi e^{-\Gamma}[a_0'^2(\delta\phi - \delta\Gamma) + 2a_0'(\delta a_0' - \delta\dot{b})] \\ & - \delta\phi''\frac{\alpha' e^\phi}{g^2 r}(1 - e^{-\Lambda}) + \delta\phi\frac{\alpha' e^\phi}{g^2 r}\left[\frac{\phi'\Lambda'}{2}(1 - 3e^{-\Lambda}) - (1 - e^{-\Lambda})(\phi'' + \phi'^2)\right];\end{aligned} \tag{4.13a}$$

$$0 = \delta\dot{\Lambda}\left[1 + \frac{\alpha' e^\phi \phi'}{2g^2 r}(1 - 3e^{-\Lambda})\right] - \frac{r\phi'}{2}\delta\dot{\phi} - \frac{\alpha' e^\phi}{g^2 r}(1 - e^{-\Lambda})\left(\delta\dot{\phi}\phi' + \delta\dot{\phi}' - \delta\dot{\phi}\frac{\Gamma'}{2}\right). \tag{4.13b}$$

Integrating Eq. (4.13b) gives, as in the previous subsection, a function $\mu(r)$ such that

$$\begin{aligned}\mu(r) = {} & \delta\Lambda\left[1 + \frac{\alpha'}{2g^2 r}e^\phi\phi'(1 - 3e^{-\Lambda})\right] - \frac{r\phi'}{2}\delta\phi \\ & - \frac{\alpha' e^\phi}{g^2 r}(1 - e^{-\Lambda})\left(\phi'\delta\phi + \delta\phi' - \frac{\Gamma'}{2}\delta\phi\right),\end{aligned} \tag{4.14}$$

which satisfies the following differential equation:

$$\begin{aligned}& \mu'(r) + \mu(r)\left[\frac{1}{2}(\Gamma' - \Lambda') + \frac{1}{r}\right] \\ & = \frac{\alpha' r}{2g^2}e^\phi e^{-\Gamma}a_0'\left[a_0'\delta\phi - \delta\dot{b} + \delta a_0' - \frac{1}{2}a_0'(\delta\Gamma + \delta\Lambda)\right].\end{aligned} \tag{4.15}$$

Since the left-hand side of Eq. (4.15) depends only on r and not on t, the same must be true of the combination of perturbations in the brackets on the right-hand side. For periodic perturbations, it follows that

$$\delta\dot{b} - \delta a_0' - a_0'\delta\phi + \frac{1}{2}a_0'(\delta\Gamma + \delta\Lambda) = 0, \tag{4.16}$$

in which case, exactly as for colored and magnetically charged black holes, $\mu(r) \equiv 0$. It should be stressed at this point that Eq. (4.16) does not in fact correspond to a choice of gauge. According to the residual gauge freedom of the gauge potential (2.5) the quantity $\delta\dot{b} - \delta a_0'$ is invariant, so we cannot choose its form. Equation (4.16) therefore represents a constraint on the gauge potential perturbations, necessary in order that Eqs. (4.13) be consistent. When $a_0 \equiv 0$ (for neutral, colored, and magnetically charged black holes) the two equations for $\delta\Lambda$ are automatically consistent, regardless of the behavior of w. The remaining gauge freedom could be used to set $\delta a_0 \equiv 0$ in the electrically charged case, as for the other types of black hole solutions.

For the construction of the Yang-Mills perturbation equations, we first define the quantity $c \equiv a_0 w$. For equilibrium black holes, c is set to vanish identically, although $w \equiv \pm 1$ and $a_0 \neq 0$ [13]. Therefore, we regard c as a separate entity, with perturbation δc, which is independent of δw and δa_0. The Yang-Mills perturbation equations then take the following form:

$$0 = \pm 2 e^{\phi} e^{(\Lambda-\Gamma)/2}(\delta c + \delta\dot{\nu}) + \partial_r \Big\{ e^{-(\Lambda+\Gamma)/2} r^2 e^{\phi} \times \Big[\delta\dot{b} - \delta a_0' - a_0' \delta\phi + \frac{1}{2} a_0' (\delta\Gamma + \delta\Lambda) \Big] \Big\}, \quad (4.17a)$$

$$0 = \pm 2 e^{\phi} e^{(\Gamma-\Lambda)/2}(\delta\nu' \pm \delta b) + e^{-(\Gamma+\Lambda)/2} r^2 e^{\phi} \times \Big[\delta\ddot{b} - \delta\dot{a}_0' - a_0' \delta\dot{\phi} + \frac{1}{2}(\delta\dot{\Gamma} + \delta\dot{\Lambda}) \Big], \quad (4.17b)$$

$$0 = e^{\Lambda-\Gamma} e^{\phi}(\delta\ddot{\nu} + \delta\dot{c}) + \partial_r [e^{(\Gamma-\Lambda)/2} e^{\phi}(\delta\nu' \pm \delta b)] + e^{\phi} e^{(\Lambda-\Gamma)/2} a_0 (a_0 \delta\nu - \delta\dot{w}), \quad (4.17c)$$

$$0 = -e^{\Lambda-\Gamma}\delta\ddot{w} + e^{\Lambda-\Gamma} a_0 \delta\dot{\nu} + \delta w'' + \Big(\phi' + \frac{\Gamma'}{2} - \frac{\Lambda'}{2} \Big) \delta w' + e^{(\Lambda-\Gamma)/2} a_0 (\delta c + \delta\dot{\nu}), \quad (4.17d)$$

where the $\pm$ depend on whether $w = \pm 1$. With the constraint (4.16), the Yang-Mills perturbation equations (4.17) simplify greatly, and reduce to the conditions

$$\delta c = -\delta\dot{\nu}, \quad \delta b = \mp \delta\nu', \quad a_0 \delta\nu = \delta\dot{w}, \quad (4.18)$$

and the equation for δw then also has a simple form

$$\delta w'' + \Big[\phi' + \frac{1}{2}(\Gamma' - \Lambda') \Big] \delta w' - \frac{2 e^{\Lambda}}{r^2} \delta w = 0. \quad (4.19)$$

It is rather surprising that this equation has no time dependence. In order to study the space dependence of δw, we are going to consider the above perturbation as the zero eigenfunction of the following eigenvalue problem:

$$\delta w'' + \Big[\phi' + \frac{1}{2}(\Gamma' - \Lambda') \Big] \delta w' - \frac{2 e^{\Lambda}}{r^2} \delta w = -\sigma^2 \delta w. \quad (4.20)$$

We note as an aside that Eq. (4.20) is the equation satisfied by periodic perturbations δw in the case of neutral equilibrium black holes. Here we shall be using Eq. (4.20) as a tool to show that there are no solutions of Eq. (4.19) representing physical perturbations. It should be emphasized that Eq. (4.19) does not restrict the time dependence of the perturbation δw at all, although the spatial dependence for each time t will be that of the solution of Eq. (4.20) when $\sigma^2 = 0$.

In order to get Eq. (4.20) in the form of a Schrödinger equation, it is convenient to define another new coordinate $\mathcal{R}$ by

$$\frac{d\mathcal{R}}{dr} = e^{-\phi} e^{-(\Gamma-\Lambda)/2}. \quad (4.21)$$

Again, the presence of the $e^{-\phi}$ term means that this is not the ''tortoise'' coordinate [compare Eqs. (3.8) and (3.16)]. Then Eq. (4.20) becomes

$$-\frac{d^2}{d\mathcal{R}^2}\delta w + \frac{2 e^{\Gamma} e^{2\phi}}{r^2} \delta w = \sigma^2 e^{2\phi} \delta w. \quad (4.22)$$

This equation is not quite of the standard Schrödinger form, due to the $e^{2\phi}$ multiplying the σ^2. However, since $e^{2\phi} > 0$, the standard theorems still apply (see, for example, [23]), so in particular there can be no negative eigenvalues σ^2 because the potential

$$U = \frac{2 e^{\Gamma} e^{2\phi}}{r^2} \quad (4.23)$$

goes to zero as $\mathcal{R} \to \pm\infty$ (i.e., at the event horizon and at infinity) and $U \geqslant 0$ everywhere. Since there are no negative eigenvalues for σ^2, this means that the solution of Eq. (4.22) with $\sigma = 0$ (if it exists) can have no zeros. For a physical perturbation δw, we require that $\delta w \to 0$ as $\mathcal{R} \to \pm\infty$. Since δw has no zeros, it must be of one sign and have at least one maximum (if it is everywhere positive) or at least one minimum (if negative). By Eq. (4.22), $d^2 \delta w / d\mathcal{R}^2$ has the same sign as δw, since the potential U is positive. Therefore, if $\delta w > 0$, at a stationary point $d^2 \delta w / d\mathcal{R}^2 > 0$ also, which means that δw has a minimum. This is in contradiction with the requirement that δw tend to zero in the asymptotic regime. A similar argument holds if δw is everywhere negative. The only possible solution of Eq. (4.19) is therefore $\delta w \equiv 0$, so that

$$\delta w = \delta\nu = \delta b = \delta c = 0. \quad (4.24)$$

We note that the fact that there are no negative eigenvalues σ^2 of Eq. (4.20) confirms that there are neutral black holes with no instabilities, even when embedded in the non-Abelian gauge group.

For electrically charged black holes we are left, as with neutral black holes, with effectively only a gravitational sector, consisting of $\delta\phi$, $\delta\Gamma$, $\delta\Lambda$, and δa_0, the latter being constrained by Eq. (4.16) to be given by

$$\delta a_0' = a_0' \delta\phi - \frac{1}{2} a_0' (\delta\Gamma - \delta\Lambda). \quad (4.25)$$

We could use the remaining gauge freedom to set $\delta a_0 \equiv 0$, but this will make no difference to our analysis. The remaining perturbation equations take the form

$$0=\delta\phi''+\delta\phi'\left(\frac{\Gamma'}{2}-\frac{\Lambda'}{2}+\frac{2}{r}\right)-\delta\phi\left[\phi''+\phi'\left(\frac{\Gamma'}{2}-\frac{\Lambda'}{2}+\frac{2}{r}\right)\right]-e^{\Lambda-\Gamma}\delta\ddot{\phi}+\frac{\alpha' e^{\phi}}{g^2r^2}(1-e^{-\Lambda})(e^{\Lambda-\Gamma}\delta\ddot{\Lambda}-\delta\Gamma'')$$
$$+\delta\Gamma'\left\{\frac{\phi'}{2}-\frac{\alpha' e^{\phi}}{g^2r^2}\left[\Lambda' e^{-\Lambda}+(1-e^{-\Lambda})\left(\Gamma'-\frac{\Lambda'}{2}\right)\right]\right\}-\delta\Lambda'\left\{\frac{\phi'}{2}-\frac{\alpha' e^{\phi}}{g^2r^2}\frac{\Gamma'}{2}(1-3e^{-\Lambda})\right\}$$
$$+\delta\Lambda\frac{\alpha' e^{\phi}}{g^2r^2}\left\{\Gamma'\Lambda' e^{-\Lambda}-e^{-\Lambda}\left[\Gamma''+\frac{\Gamma'}{2}(\Gamma'-\Lambda')\right]\right\}+\frac{\alpha'}{2g^2}e^{\phi}e^{-\Gamma}[3a_0'^2\delta\phi-2a_0'^2\delta\Gamma-a_0'^2\delta\Lambda], \tag{4.26a}$$

$$0=\delta\Gamma'\left[1+\frac{\alpha' e^{\phi}\phi'}{2g^2r}(1-3e^{-\Lambda})\right]+\delta\phi'\left[-\frac{r\phi'}{2}+\frac{\alpha' e^{\phi}}{2g^2r}\Gamma'(1-3e^{-\Lambda})\right]-\delta\ddot{\phi}\frac{\alpha' e^{\phi}}{g^2r}e^{\Lambda-\Gamma}(1-e^{-\Lambda})$$
$$+\delta\Lambda\left[-\frac{e^{\Lambda}}{r}+\frac{3\alpha' e^{\phi}}{2g^2r}\phi'\Lambda' e^{-\Lambda}\right]+\delta\phi\frac{\alpha' e^{\phi}}{2g^2r}\phi'\Gamma'(1-3e^{-\Lambda})+\frac{\alpha' r}{4g^2}e^{\phi}e^{-\Gamma}[3a_0'^2\delta\phi-2a_0'^2\delta\Gamma-a_0'^2\delta\Lambda], \tag{4.26b}$$

$$0=\delta\Gamma''\left(1-\frac{\alpha' e^{\phi-\Lambda}}{g^2r}\phi'\right)+\frac{\delta\Gamma'}{2}(\Gamma'-\Lambda')+(\delta\Gamma'-\delta\Lambda')\left(\frac{\Gamma'}{2}+\frac{1}{r}\right)-e^{\Lambda-\Gamma}\delta\ddot{\Lambda}$$
$$-\frac{\alpha' e^{\phi-\Lambda}}{g^2r}\left\{(\delta\phi''+2\phi'\delta\phi')\Gamma'+\delta\phi'\Gamma''+\delta\Gamma'(\phi''+\phi'^2)+\left(\frac{\delta\phi'\Gamma'}{2}+\frac{\phi'\delta\Gamma'}{2}\right)(\Gamma'-3\Lambda')\right.$$
$$\left.+(\delta\phi-\delta\Lambda)\left[\phi'\Gamma''+\Gamma'(\phi''+\phi'^2)+\frac{\phi'\Gamma'}{2}(\Gamma'-3\Lambda')\right]+\frac{\phi'\Gamma'}{2}(\delta\Gamma'-3\delta\Lambda')+e^{\Lambda-\Gamma}(\Lambda'\delta\ddot{\phi}-\phi'\delta\ddot{\Lambda})\right\}$$
$$+\phi'\delta\phi'-\frac{\alpha'}{2g^2}e^{\phi}e^{-\Gamma}[3a_0'^2\delta\phi-2a_0'^2\delta\Gamma-2a_0'^2\delta\Lambda], \tag{4.26c}$$

where $\delta\Lambda$ is given by Eq. (4.14). This could be reduced to a single equation by using Eq. (4.14) and (4.25). However, since we shall be appealing to catastrophe theory, this does not afford any advantage in the analysis.

Now that the system has been reduced to a set of coupled equations (4.26) involving perturbations of the dilaton field and metric functions, we can once again appeal to the catastrophe theory analysis of [13]. They find two branches of electrically charged black hole solutions, one of which extends out to an infinite horizon radius. We shall focus on this branch of solutions since it is this branch (if any) which can be stable. As in the previous section, we consider perturbation equations in the limit $r_h\to\infty$, and work to leading order in $1/r_h^2$. Using the same tortoise coordinate $\hat{r}^*$ as in the previous subsection, we find the equations

$$0=-\delta\ddot{\phi}+\frac{d^2}{d\hat{r}^{*2}}\delta\phi+\frac{2}{\hat{r}}e^{(\Gamma-\Lambda)/2}\frac{d}{d\hat{r}^*}\delta\phi-\left[\frac{d^2\phi}{d\hat{r}^{*2}}+\frac{2}{\hat{r}}e^{(\Gamma-\Lambda)/2}\frac{d\phi}{d\hat{r}^*}\right]\delta\phi+\frac{1}{2}\frac{d\phi}{d\hat{r}^*}\left(\frac{d}{d\hat{r}^*}\delta\Gamma-\frac{d}{d\hat{r}^*}\delta\Lambda\right)+O\left(\frac{1}{r_h^2}\right),$$

$$0=e^{(\Gamma-\Lambda)/2}\frac{d}{d\hat{r}^*}\delta\Gamma-\frac{\hat{r}}{2}\frac{d\phi}{d\hat{r}^*}\frac{d}{d\hat{r}^*}\delta\phi-\frac{e^{\Gamma}}{\hat{r}}\delta\Lambda+O\left(\frac{1}{r_h^2}\right), \tag{4.27}$$

$$0=\delta\Lambda-\frac{\hat{r}}{2}\frac{d\phi}{d\hat{r}^*}\delta\phi+O\left(\frac{1}{r_h^2}\right).$$

These are exactly the same equations (without the equation for δw) as those obtained for the other types of black hole solutions (4.5). In this case, for $r_h\gg 1$, the geometry reduces to a Reissner-Nordström black hole, with $\phi\equiv$const. As in the Schwarzschild case, the perturbations $\delta\Gamma$ and $\delta\Lambda$ vanish identically, and the equation for $\delta\phi$ reduces to Eq. (4.6). The same argument then shows that there are no instabilities in this sector for $r_h\gg 1$. The catastrophe theory analysis then tells us that the upper branch of electrically charged black holes is stable for all r_h.

We conclude in this subsection that the upper branch of electrically charged black holes is stable, whereas we have already shown that magnetically charged black holes are infinitely unstable. This may be surprising, especially since the gauge field in both cases is essentially Abelian. However, for magnetically charged black holes, the gauge potential is *embedded Abelian*; in other words, it corresponds to the product of a U(1) gauge potential and a constant matrix. This embedding in the non-Abelian gauge group SU(2) gives rise to an infinite number of unstable modes. As discussed earlier in this article, the same conclusion can be reached from observing that magnetically charged black holes arise as the limit of colored black holes in which the number of nodes of the gauge function w goes to infinity. This leads to infinitely many modes of instability in the sphaleronic sector. On the other hand, because of the construction of electrically charged black holes (in particular, setting $c \equiv a_0 w \equiv 0$ in the field equations although neither a_0 nor w vanish) means that the gauge potential is genuinely U(1) (without any embedding). Therefore the stability of these solutions is not unexpected. In this case there is effectively no sphaleronic sector, so other instabilities present for colored and magnetically charged black holes do not arise.

V. CONCLUSIONS

In this article, we have considered a generalized, string-inspired theory of gravity that describes the nonminimal coupling of a single scalar field, the dilaton, to gravity through the higher-derivative Gauss-Bonnet term as well as to a non-Abelian SU(2) gauge field. This theory has been shown in previous work [12] to admit regular and asymptotically flat black hole solutions that are characterized by the presence of a nontrivial dilaton and gauge field on the region outside the horizon in contradiction with the ''no-hair'' theorem of the theory of general relativity. Nevertheless, the hair of these black hole solutions is merely ''secondary'' in the sense that no new charges can be associated with the aforementioned nonvanishing fields. The dilatonic black holes that arise in the same framework but in the absence of the gauge field [15] have been already proved [17] to be linearly stable under small, bounded, spacetime-dependent perturbations as they correspond to the stable branch of the family of neutral black hole solutions. Then, the question of the behavior of the colored black holes under the same type of perturbations, and in the presence of the same stringy corrections, naturally arises.

By making an appropriate choice of gauge, the linearized perturbation equations, for the colored black holes, were decoupled into two sectors, the sphaleronic and gravitational ones. In the first sector, the perturbation equations resembled those of Einstein-Yang-Mills-dilaton theory with the explicit dependence on the Gauss-Bonnet term having been eliminated. For the needs of our analysis, well-known methods [6,7], which were previously used for the stability analysis of black hole solutions arising in the framework of Einstein-Yang-Mills theory, were extended in order to accommodate the dilaton field. Then, the existence of topological instabilities was analytically demonstrated by making use of the method of trial functions. The number of unstable modes in this sector was determined by mapping the irregular Schrödinger equation of the gauge perturbations to a ''dual'' regular one and it was found to be the same as the number of zeros of the background gauge field. In the gravitational sector, as a result of the complexity of the perturbation equations, continuity arguments based on the results derived from the catastrophe theory analysis [13] allowed us to work in the limit of an infinitely large horizon value. Although the subsystem of the metric and dilaton perturbations was found to be stable, instabilities attributed once again to the oscillating behavior of the background gauge function around zero were proved to exist. As a result, we may conclude that the accommodation of stringy corrections to non-Abelian black holes fails to render these solutions linearly stable.

The perturbation equations for magnetically charged and neutral black holes, under the same type of perturbations, may easily follow from the corresponding ones for the colored black holes by simply choosing an appropriate value for the background gauge function w. This allows us to draw conclusions concerning the stability of these two families of solutions in the same framework and by using the same methods as above. Then, the study of the sphaleronic sector reveals the existence of an infinite number of unstable modes, for magnetically charged black holes, and the same holds for the gravitational sector. The above result is in accordance with the interpretation of this family of solutions as the limit of colored black holes in which the number of zeros of w goes to infinity. On the other hand, no unstable modes are found for the upper branch of neutral black holes in both sectors which confirms the stable character of these solutions not only under metric and scalar perturbations [17] but even under gauge-dependent perturbations.

Finally, the stability analysis of the electrically charged solutions was conducted although in a different framework of perturbation equations due to the different *Ansätze* for the background gauge field. Nevertheless, we were able to show that, in order for our perturbation equations to be consistent, a constraint that involves a combination of gauge, metric, and scalar perturbations must be satisfied. In that case, we have shown that the gauge perturbations are completely decoupled from the scalar and metric ones, a feature which facilitates the study of each subsector. The gauge perturbations were found to reduce to a single equation for δw which, however, was shown not to accept any solutions representing physical perturbations. On the other hand, by using the same continuity arguments and working again in the limit of an infinitely large horizon, we proved that no unstable modes arise in the remaining subsector of metric and scalar perturbations for the upper branch of the electrically charged black hole solutions. In conclusion, the bulk of electrically charged black holes, in common with neutral dilatonic black holes, are stable under small, spacetime-dependent perturbations. The stability of these two families of solutions can be justified by their interpretation as a generalization of Reissner-Nordström and Schwarzschild black holes, respectively, in the framework of string theory.

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